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CHARACTERIZATION OF NONHOMOGENEOUS POISSON PROCESSES
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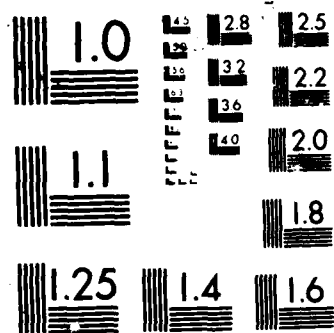
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$m(t) = EN(t)$ if and only if for any $t_i, i = 1, \dots, k$

$$\text{cum}(N(t_1), \dots, N(t_k)) = \min_{1 \leq i \leq k} EN(t_i)$$

where $\text{cum}(\cdot)$ is the joint multivariate cumulant.

A second result is that if increments on any interval are Poisson distributed and an exchangeable condition is assumed then the process is nonhomogeneous Poisson. This extends Renyi's (1967) result.

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CHARACTERIZATION OF NONHOMOGENEOUS POISSON PROCESSES VIA MOMENT CONDITONS

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Abstract

The property of independent increments is one of the most important for defining both the homogeneous and nonhomogeneous Poisson process.

In this paper we give two ways to relax this requirement and characterize the nonhomogeneous Poisson process by some moment conditons.

One result is that a counting process $\{N(t), t \geq 0\}$ with finite moments of all orders is a nonhomogeneous Poisson process with mean functions $m(t) = EN(t)$ if and only if for any $t_i, i = 1, \dots, k$

$$\text{cum}(N(t_1), \dots, N(t_k)) = \min_{1 \leq i \leq k} EN(t_i)$$

where $\text{cum}(\cdot)$ is the joint multivariate cumulant.

A second result is that if increments on any interval are Poisson distributed and an exchangeable condition is assumed then the process is nonhomogeneous Poisson. This extends Renyi's (1967) result.

Key Words: Nonhomogeneous Poisson Process, independent increments, characterization problem, joint cumulant, exchangeable.

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1. Introduction

Poisson processes play an important role in many fields. The Poisson process is one of the simplest counting processes and is a building block for many other processes, especially for general independent increment processes.

Many definitions have been given for a counting processes to be a Poisson process and many papers have appeared dealing with characterizations for the Poisson distribution and the Poisson process.

Among the qualitative conditions defining a Poisson process, independent increments is one of the most important conditions. In this paper we attempt to use other conditions in place of independent increments. This provides a somewhat different viewpoint for examining Poisson processes. In addition, new characterizations for the nonhomogeneous Poisson process via moment conditions are obtained which might be easier to utilize in practice.

For clarity we give a standard definition first.

Definition 1. A counting process $\{N(t), t \geq 0\}$ is said to be a nonhomogeneous Poisson process with parameter function $m(t)$ if

- (i) $P\{N(0)=0\} = 1$,
- (ii) the process has independent increments,
- (iii) for $0 \leq s < t$, $N(t)-N(s)$ is Poisson distributed with mean $m(t)-m(s)$, i.e.

$$P\{N(t)-N(s)=k\} = \frac{\{m(t)-m(s)\}^k}{k!} \exp \{-(m(t)-m(s))\} \quad (1)$$

for $k=0,1,2,\dots$ where $m(t)$ is a nonnegative nondecreasing function of t .

For the purpose of exposition we assume $m(t)$ is continuous. Renyi (1967) made an essential improvement in relaxing the condition of independent increments. This is given in the Theorem 1 below.

Let $R^+ = \{t, t \geq 0\}$. B be the Borel set of R^+ , $\mathcal{J} = \{\text{finite union of disjoint finite intervals } (a,b] \text{ on } R^+\}$, and $I=(s,t]$ be an interval on R^+ . Let $\xi(E)$ be an additive stochastic set function defined for $E \in \mathcal{J}$ which counts the number of events of a counting process $N(t)$ falling in E . In other words, $\xi(E)$ represents the total increments of the process $N(t)$ on intervals in E . It follows that $\xi(I) = N(t) - N(s)$ is the increment on $(s,t]$. Let $\lambda(E)$ be a measure with no atoms in $E \in \mathcal{J}$, such that $\lambda(E) < \infty$ for $E \in \mathcal{J}$ and $\lambda(I) = m(t) - m(s)$ where $I = (s,t]$. Here $m(t)$ is a nonnegative nondecreasing function of t which is assumed to be continuous.

Theorem 1. Let $\{N(t), t \geq 0\}$ be a counting process with $P\{N(0)=0\}=1$. If for $E \in \mathcal{J}$ $\xi(E)$ has a Poisson distribution with mean value $\lambda(E)$, then $N(t)$ is a Poisson process with mean $m(t)$.

Proof. Renyi (1967).

If it is assumed only that $\xi(I)$ has a Poisson distribution for any interval I in R^+ , Shepp (see Goldman, 1968) has constructed a counter-example to show that the process need not be Poisson. Furthermore, Oakes (1972) has provided a construction of a counting process such that even if for fixed k the counts $\xi(I_i)$ in any k contiguous interval $I_i = (t_{i-1}, t_i]$ are independently Poisson distributed with mean $t_i - t_{i-1}$, $N(t)$ need not be a Poisson process. This process is called a k -fold quasi-Poisson process.

An interesting question is to determine additional conditions needed to ensure that a process is nonhomogeneous Poisson with mean $m(t)$ besides the condition that $\xi(I)$ is distributed as Poisson with mean $\lambda(I)$. In Section 2, we introduce an exchangeable condition which was suggested by Dr. Z. D. Bai (personal communication) to replace the independence condition.

It is well known that independence implies uncorrelatedness, but the converse is not true in general. In Section 3, we use the idea of joint cumulants developed in Block and Fang (1985) to give a

characterization for nonhomogeneous Poisson processes via moment conditions under which uncorrelatedness between increments implies independent increments. It naturally follows that if $\xi(I)$ is Poisson distributed for any interval then the counting process is nonhomogeneous Poisson.

Both arguments above relax the common condition of independent increments for a process to be Poisson.

2. An exchangeability condition

In this section we prove that a counting process is Poisson under an exchangeability condition. Here we use exchangeability with respect to a finite number of rv's.

Consider a counting process $N(t)$ on R^+ . Let I be the interval $(a,b]$, $|I| = b-a$ (the Lebesgue measure for interval I). $A(I) = \{N(b)-N(a)=0\}$ (the event that there is no occurrence of the process in I), and let $1_{A(I)}$ be the indicator function of $A(I)$.

First we prove a lemma to show how the exchangeable condition works, then by using this lemma we obtain a new set of conditions for a Poisson process.

Lemma 1. If for any interval $I = (a,b]$ on R^+

$$P\{N(b)-N(a) = 0\} = e^{-\lambda|I|} \quad (2)$$

and for any contiguous intervals $I_i = (t_{i-1}, t_i]$ $i=1, \dots, k$ such that $|I_i| \equiv \delta$, $\delta > 0$, the rv's $1_{A(I_i)}$, $i=1, \dots, k$ are exchangeable, then $1_{A(I_i)}$ $i=1, \dots, k$ are independent.

Proof. From $A(I_i) = \{N(t_i)-N(t_{i-1}) = 0\}$ we know that if $J = \bigcup_{m=1}^M I_{i_m}$, $i_m \in \{1, \dots, k\}$ $m = 1, \dots, M$

then $A(J) = \bigcap_{m=1}^M A(I_{i_m})$ and $1_{A(J)} = \prod_{m=1}^M 1_{A(I_{i_m})}$. It is easy to see that the $1_{A(I_i)}$ are identically distributed Bernoulli rv's with parameter p , where $p = e^{-\lambda\delta}$.

Similarly, since the $1_{A(I_i)}$'s are exchangeable we have

$$P\{1_{A(J)} = 1\} = P\{1_{A(I_{i_m})} = 1, m = 1, \dots, M\}$$

$$= P\{1_{A(I_i)} = 1, i = 1, \dots, M\}$$

$$= P\{N(t_M) - N(t_0) = 0\}$$

$$= e^{-\lambda|(t_M - t_0)|} = e^{-\lambda(t_M - t_0)} = e^{-\lambda \delta M} = p^M$$

i.e., $1_{A(I)}$ is Bernoulli with parameter p^M .

Now, for $f_i \in \{0, 1\}$, $i = 1, \dots, k$. Let $B_0 = \{i | f_i = 0\}$, $B_1 = \{i | f_i = 1\}$. $B_0 \cup B_1 = \{1, \dots, k\}$.
Card $B_0 = M_0$, Card $B_1 = M_1$ and $M_0 + M_1 = k$. Denote the complement of $A(I)$ by $A^c(I)$. By exchangeability we have

$$\begin{aligned} & P\{1_{A(I_i)} = f_i, i = 1, \dots, k\} \\ &= P\{1_{A(I_i)} = 0, i \in B_0; 1_{A(I_i)} = 1, i \in B_1\} \\ &= P\{1_{A(I_i)} = 0, i = 1, \dots, M_0; 1_{A(I_i)} = 1, i = M_0 + 1, \dots, k\} \\ &= P\{1_{A^c(I_i)} = 1, i = 1, \dots, M_0; 1_{A(I_i)} = 1, i = M_0 + 1, \dots, k\} \\ &= E \left\{ \prod_{i=1}^{M_0} 1_{A^c(I_i)} \prod_{i=M_0+1}^k 1_{A(I_i)} \right\} \\ &= E \left\{ 1_{\bigcap_{i=1}^{M_0} A^c(I_i)} 1_{\bigcap_{i=M_0+1}^k A(I_i)} \right\} \\ &= E \left\{ \left(1 - 1_{\bigcup_{i=1}^{M_0} A(I_i)} \right) 1_{\bigcap_{i=M_0+1}^k A(I_i)} \right\} \\ &= E \left\{ 1_{\bigcap_{i=M_0+1}^k A(I_i)} \right\} - E \left\{ \sum_{j=1}^{M_0} (-1)^{j-1} S_j \prod_{i=M_0+1}^k 1_{A(I_i)} \right\} \end{aligned} \quad (3)$$

where $S_j = \sum_{1 \leq i_1 < \dots < i_j \leq M_0} 1_{A(I_{i_1})} \cap \dots \cap A(I_{i_j})$. Since the $1_{A(I_i)}$'s are exchangeable

$$\begin{aligned}
 & E \left\{ S_j \prod_{i=M_0+1}^k 1_{A(I_i)} \right\} \\
 &= \sum_{1 \leq i_1 < \dots < i_j \leq M_0} E 1_{\bigcap_{m=1}^j A(I_{i_m})} \prod_{i=M_0+1}^k 1_{A(I_i)} \\
 &= \sum_{1 \leq i_1 < \dots < i_j \leq M_0} E 1_{A(I_{i_1})} \cap \dots \cap A(I_{i_j}) \cap A(I_{M_0+1}) \cap \dots \cap A(I_k) \\
 &= \sum_{1 \leq i_1 < \dots < i_j \leq M_0} p^{k-M_0+j} = \binom{M_0}{j} p^j p^{k-M_0} \quad (4)
 \end{aligned}$$

It follows from (3) and (4) that

$$\begin{aligned}
 & P \left\{ 1_{A(I_i)} = f_i, i = 1, \dots, k \right\} \\
 &= p^{k-M_0} - \sum_{j=1}^{M_0} (-1)^{j-1} \binom{M_0}{j} p^j p^{k-M_0} \\
 &= p^{k-M_0} (1-p)^{M_0} = \prod_{i=1}^k P \left\{ 1_{A(I_i)} = f_i \right\}
 \end{aligned}$$

We now state the main result of this section. Here the usual condition of independent increments is replaced by an exchangeability condition.

Theorem 2. Let $\{N(t) \ t \geq 0\}$ be a counting process. If the conditions i) - iii) below hold, the $\{N(t), t \geq 0\}$ is a homogeneous Poisson process with mean function $E\{N(t)\} = \lambda t$.

i) For any interval $I = (a, b]$ $P\{N(b) - N(a) = 0\} = e^{-\lambda |I|}$.

- ii) $P\{N(b) - N(a) \geq 2\} = o(\lambda|I|)$ as $|I| \rightarrow 0$ uniformly in I
- iii) For any finite number of contiguous intervals $I_i = (t_{i-1}, t_i]$ $i = 1, \dots, k$ such that $|I_i| \equiv \delta$, the rv's $1_{A(I_i)}$ $i = 1, \dots, k$ are exchangeable.

Proof. By the proof of Theorem 1 and Remark 1 in Renyi (1967) we need only prove that for any disjoint intervals E_i , $i = 1, \dots, n$ $1_{A(E_i)}$ $i = 1, \dots, n$ are independent. If this is proven, it then follows from the argument there that $\{N(t) : t \geq 0\}$ is a process with independent increments and consequently $\{N(t) : t \geq 0\}$ is Poisson process.

We can arrange the E_i 's according to their order in R^+ , relabeling them if necessary, and cut them into small intervals of length ε beginning at E_1 . For any $\varepsilon > 0$ there exist a set of intervals E'_i $i = 1, \dots, n$ such that $E'_i \subset E_i$, $|E'_i| = c_i \varepsilon$ c_i are integers and $|E_i| - |E'_i| \leq 2\varepsilon$. Furthermore, the E_i can be chosen so that F_i the space between E'_i and E'_{i+1} are intervals of length $|F_i| = d_i \varepsilon$, where d_i are also integers, $i = 1, \dots, n-1$ as shown in the diagram. Denote the interval of length ε which contains $F_{i-1} \cap E_i$ by Δ_{i1} and the interval of length ε which contains $F_i \cap E_i$ by Δ_{i2} . When $F_{i-1} \cap E_i$ or $F_i \cap E_i$ is empty, define $\Delta_{ij} = \text{empty}$. Thus $|\Delta_{ij}| \leq \varepsilon$, $j = 1, 2$, $i = 1, \dots, n$.

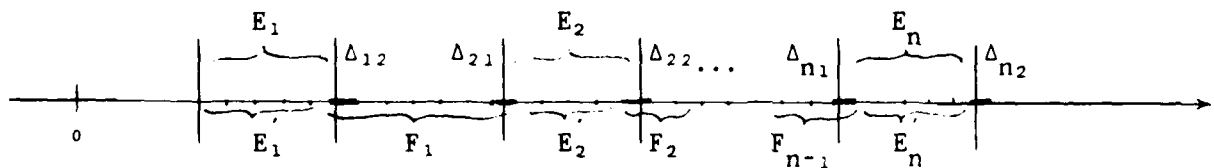


Diagram 1

By Lemma 1, all contiguous subintervals I_q of length ε have the property that $1_{A(I_q)}$'s are independent. It follows that $1_{A(E'_i)}$'s are independent since $E'_i \cap E'_j = \text{empty}$ for $i \neq j$ and $1_{A(E'_i)}$ only depend on $1_{A(I_q)}$'s such that $I_q \subset E'_i$. Since $E'_i \subset E_i \subset \Delta_{i1} \cup E'_i \cup \Delta_{i2}$, we have for any

$$1 \leq i_1, \dots, i_m \leq n \quad m = 1, \dots, n$$

$$\begin{aligned} & E \, 1_{A(E_{i_1})} \cdots 1_{A(E_{i_m})} \\ & \geq E \, 1_{A(E_{i_1})} \cdots 1_{A(E_{i_m})} \\ & \geq E \prod_{j=1}^m 1_{A(\Delta_{i_j,1}) \cup E_{i_j} \cup \Delta_{i_j,2}} \\ & = E \left\{ \prod_{j=1}^m 1_{A(E_{i_j})} \prod_{j=1}^m [1_{A(\Delta_{i_j,1})} 1_{A(\Delta_{i_j,2})}] \right\} \\ & = E \left\{ \prod_{j=1}^m 1_{A(E_{i_j})} \right\} E \left\{ \prod_{j=1}^m 1_{A(\Delta_{i_j,1})} \right\} E \left\{ \prod_{j=1}^m 1_{A(\Delta_{i_j,2})} \right\} \\ & = E \left\{ \prod_{j=1}^m 1_{A(E_{i_j})} \right\} e^{-2\lambda m \varepsilon} \end{aligned}$$

Thus

$$\begin{aligned} & E \left\{ \prod_{j=1}^m 1_{A(E_{i_j})} \right\} e^{-2\lambda m \varepsilon} - \prod_{j=1}^m E \, 1_{A(E_{i_j})} \\ & \leq E \prod_{j=1}^m 1_{A(E_{i_j})} - \prod_{j=1}^m E \, 1_{A(E_{i_j})} \\ & \leq E \prod_{j=1}^m 1_{A(E_{i_j})} - \left\{ \prod_{j=1}^m E \, 1_{A(E_{i_j})} \right\} e^{-2\lambda m \varepsilon} \end{aligned}$$

Since ε are arbitrary, we obtain that the $1_{A(E_{i_j})}$'s are independent from the fact that $1_{A(E_{i_j})}$'s are independent.

For the nonhomogeneous case, we have similar results.

Theorem 3 Let $\{N(t) : t \geq 0\}$ be a counting process and let λ be a nonatomic measure on B the class of Borel sets on R^+ . For intervals $I = (a, b] \subset R^+$, let $\lambda(I) = m(b) - m(a)$, where m is a nonnegative nondecreasing continuous function with $m(0) = 0$. If the following conditions i)-iii) are satisfied, then $N(t)$ is a nonhomogeneous Poisson process with mean function $E(N(t)) = m(t)$.

- i) For any interval $I = (a, b]$, $P\{N(b) - N(a) = 0\} = e^{-\lambda(I)}$
- ii) $P\{N(b) - N(a) = 0\} \geq 2\} = o(\lambda(I))$ as $\lambda(I) \rightarrow 0$ uniformly in I .
- iii) For any finite number of contiguous intervals $I_i = (t_{i-1}, t_i]$ $i=1, \dots, k$ such that $\lambda(I_i) \equiv \delta_i$, $\delta_i > 0$, $i=1, \dots, k$, the rv's $1_{A(I_i)}$ are exchangeable.

Proof We follow the same line of argument as in Lemma 1 and Theorem 2. We need only replace $p = e^{-\lambda\delta}$ by $p = e^{-\delta_i}$, divide the E_i 's by sub intervals I_i with $\lambda(I_i) \equiv \delta_i$, etc. We omit the details.

Remark 1 Without i) and ii) in Theorem 3, the condition of independent increments and the exchangeability of the $1_{A(I_i)}$ in iii) of Theorem 3 are not equivalent.

Remark 2 Definition 1, the condition in Theorem 1 (i.e., $\xi(E)$ is Poisson distributed) and the conditions in Theorem 3 are equivalent. We show this in the following.

Definition 1 implies the conditions in Theorem 3, i.e., if $N(t)$ has independent increments then

$$E 1_{A(I_1)} \cdots 1_{A(I_m)} = E 1_{A(I_1)} \cdots E 1_{A(I_m)} = e^{-m\delta_1} = E 1_{A(I_1)} \cdots 1_{A(I_m)}$$

for any $1 \leq i_1, \dots, i_m \leq k$ so that $1_{A(I_i)}$'s are exchangeable.

The conditions in Theorem 1 imply the conditions in Theorem 3, i.e., let $E = \bigcup_{j=1}^m I_j$, $E \in \mathcal{E}$. Then

$$\begin{aligned} & E 1_{A(I_1)} \cdots 1_{A(I_m)} \\ &= P(\xi(I_j) = 0, j = 1, \dots, m) \\ &= P(\xi(E) = 0) = e^{-\lambda(E)} = e^{-m\delta_1} \end{aligned}$$

$$= E 1_{A(I_1)} \cdots 1_{A(I_m)}$$

for any $1 \leq i_1, \dots, i_m \leq k$. Thus the $1_{A(I_i)}$'s are exchangeable.

By Theorem 3 we know that these three sets of conditions are equivalent. However, the conditions in Theorem 3 are somewhat easier to check.

Remark 3. Oakes (1972) mentioned the result that if $\xi(I_i)$, $i = 1, \dots, m$, where the I_i are contiguous intervals, are independently distributed as a Poisson distribution with mean $|I_i|$ for all m , then $N(t)$ must be Poisson. This fact also follows from Theorem 2. To see this choose special contiguous intervals I_i such that $|I_i| \equiv \delta$ then $\xi(I_i)$ are i.i.d Poisson (δ). Thus the $1_{A(I_i)}$'s are exchangeable for any I_i $i = 1, \dots, m$ of length δ and contiguous. By Theorem 2 $N(t)$ is a Poisson process.

Remark 4. Under condition iii) of Theorem 2 or Theorem 3 we can weaken conditions i) and ii). We need only assume they hold for the special class of intervals having the form $I = (0, t]$.

3. Characterization Via Moments

We recall some definitions and properties for the r th joint cumulant which was discussed in a recent paper by Block and Fang (1985).

Definition 2. The r th joint cumulant of (X_1, \dots, X_r) is defined by

$$\text{cum}(X_1, \dots, X_r) = \sum (-1)^{p-1} (p-1)! \left[E \prod_{j \in v_1} X_j \right] \cdots \left[E \prod_{j \in v_p} X_j \right] \quad (5)$$

where the summation extends over all partitions (v_1, \dots, v_p) , $p = 1, 2, \dots, r$, of $\{1, \dots, r\}$.

We state some properties which are easy to check.

- (i) If any group of the X 's are independent of the remaining X 's then $\text{Cum}(X_1, \dots, X_r) = 0$.
- (ii) For (X_1, \dots, X_r) independent of (Y_1, \dots, Y_r) $\text{Cum}(X_1 + Y_1, \dots, X_r + Y_r) = \text{Cum}(X_1, \dots, X_r) + \text{Cum}(Y_1, \dots, Y_r)$.

(iii) $\text{Cum } X_i = EX_i$ and $\text{Cum } (X_i, X_j) = \text{Cov } (X_i, X_j)$.

(iv) The joint cumulant is a multilinear operator.

A useful relationship between moments and cumulants is given in the following lemma.

Lemma 2. If $E|X_i|^m < \infty$ for any i

$$EX_1 \cdots X_m - EX_1 \cdots EX_m = \sum \text{cum } (X_k, k \in v_1) \cdots \text{cum } (X_k, k \in v_p) \quad (6)$$

where \sum extends over all partitions (v_1, \dots, v_p) , $p = 1, \dots, m-1$ of $\{1, \dots, m\}$.

Proof. See Lemma 1 of Block and Fang (1985).

For the characterization problem here we need the following result.

Lemma 3. If for all positive integers $q_i \leq r_i$ $i = 1, 2, \dots, m$ such that $\sum q_i < \sum r_i$ we have

$$\begin{aligned} EY_1^{q_1} \cdots Y_m^{q_m} &= EY_1^{q_1} \cdots EY_m^{q_m}, \text{ then } EY_1^{r_1} \cdots Y_m^{r_m} - EY_1^{r_1} \cdots EY_m^{r_m} \\ &= \text{cum } (\underbrace{Y_1, \dots, Y_1}_{r_1}, \dots, \underbrace{Y_m, \dots, Y_m}_{r_m}) \end{aligned} \quad (7)$$

Proof. By Lemma 2.

$$\begin{aligned} EY_1^{r_1} \cdots Y_m^{r_m} - EY_1^{r_1} \cdots EY_m^{r_m} \\ = \sum \text{cum } (Y_1^{r_1}, i \in v_1) \cdots \text{cum } (Y_1^{r_1}, i \in v_p) \end{aligned} \quad (8)$$

where the sum is overall partition (v_1, \dots, v_p) $p = 1, 2, \dots, m-1$ of $\{1, \dots, m\}$.

Now by the conditions of Lemma 3 and induction we have the previous sum is

$$\text{cum } (Y_1^{r_1}, \dots, Y_m^{r_m}). \quad (9)$$

Using Theorem 2.3.2. of Brillinger (1975) this turns out to be

$$\sum \text{cum} (Y_j, j \in v_1) \cdots \text{cum} (Y_j, j \in v_p) \quad (10)$$

where the summation is over all indecomposable partitions of the set $\{1, \dots, 1, \dots, m, \dots, m\}$ wrt to partition $\{1, \dots, 1\}, \dots, \{m, \dots, m\}$ where the number of i 's is r_i for $i = 1, \dots, m$. The desired result follows by using the conditions of Lemma 3 and induction once again. Since for all proper indecomposable partitions v_i , $\text{cum} (Y_j, j \in v_i) = 0$.

Now we return to the nonhomogeneous Poisson processes. Some basic facts are reviewed below for the nonhomogeneous Poisson process $\{N(t), t \geq 0\}$ with mean $m(t) = EN(t)$, $t \geq 0$.

(i) The characteristic function of $N(t)$

$$\phi_{N(t)}(u) = E\{\exp(iuN(t))\} = \exp\{m(t)(e^{iu}-1)\} \quad (11)$$

is analytic in the whole plane.

(ii) The m th moments of $N(t)$ are polynomial in $\mu = m(t)$. More precisely we have

$$E\{(N(t))^k\} = \sum_{i=0}^k S_{i,k} \mu^i \quad (12)$$

where $S_{i,k}$ is the Stirling numbers of the second kind defined by formula

$$S_{i,k} = \frac{1}{i!} \sum_{j=0}^i (-1)^{i-j} \binom{i}{j} j^k \quad (13)$$

(See Riordan, 1937)

We characterize the nonhomogeneous Poisson process as follows:

Theorem 4. Let $\{N(t), t \geq 0\}$ be a counting process with finite moments of all orders. This process is a nonhomogeneous Poisson process if and only if for all real t_i , $i=1, \dots, k$

$$\text{cum} (N(t_1), \dots, N(t_k)) = \min_{1 \leq i \leq k} EN(t_i) . \quad (14)$$

Proof. If $\{N(t), t \geq 0\}$ is a nonhomogeneous Poisson process, we assume $t_1 \leq t_2 \leq \dots \leq t_k$. Since $N(t)$

has independent increments, $N(t_1)$ is independent of $N(t_i) - N(t_1)$, $i = 2, \dots, k$. Thus $(N(t_1), \dots, N(t_k))$ is independent of $(0, N(t_2) - N(t_1), \dots, N(t_k) - N(t_1))$. Using properties of the cumulant

$$\begin{aligned} & \text{cum}(N(t_1), \dots, N(t_k)) \\ &= \text{cum}(N(t_1), N(t_1) + N(t_2) - N(t_1), \dots, N(t_1) + N(t_k) - N(t_1)) \\ &= \text{cum}(N(t_1), \dots, N(t_1)) \\ &+ \text{cum}(0, N(t_2) - N(t_1), \dots, N(t_k) - N(t_1)) \\ &= \text{cum}(N(t_1), \dots, N(t_1)) \end{aligned} \quad (15)$$

Let $C_{i,k}$ denote the number of ways of partitioning $\{1, \dots, k\}$ into i groups for where $i > k$ or $i < 1$ we let $C_{i,k} = 0$. For example when $k=2$ we have $C_{0,2} = 0$, $C_{1,2} = 1$, $C_{2,2} = 1$, $C_{3,2} = 0$ etc. For all k , $C_{1,k} \equiv 1$, $C_{k,k} \equiv 1$. We also have following recurrence formula.

$$C_{i,k+1} = i C_{i,k} + C_{i-1,k} \quad (16)$$

This result follows since one can count the number of ways of partitioning $\{1, \dots, k+1\}$ into i groups by considering two cases. One case is obtained by setting one element aside, then partitioning the remaining k elements into i groups and then replacing the single element in any of these groups. The total number is $iC_{i,k}$. The second case is obtained by counting the numbers of partitioning into i groups where the single element forms one group and then partition the remaining k elements into $i-1$ groups. This number is $C_{i-1,k}$.

Notice that (15) coincides with the recurrence formula for $S_{i,k}$ in (12) and

$$\begin{aligned} S_{0,k} &= 0 = C_{0,k} \\ S_{1,k} &= (-1)^{1-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix} 1^k \\ &= 1 = C_{1,k} \end{aligned}$$

So we get

$$S_{i,k} = C_{i,k} \quad (17)$$

Now using this fact, we can prove that $\text{cum}(N(t_1), \dots, N(t_1)) = \mu$, where $\mu = EN(t_1)$. We use induction on k , the number of $N(t_i)$'s in $\text{cum}(N(t_1), \dots, N(t_1))$. When $k=1$ the result is trivial. For

$k=2$, $\text{Cov}(N(t_1), N(t_1)) = \text{Var } N(t_1) = \mu$, which is well known. For $k > 2$, by Lemma 2 we obtain

$$\begin{aligned} & \text{cum}(N(t_1), \dots, N(t_1)) \\ &= E\{N(t_1)^k\} - \sum \text{cum}(N(t_1), i\varepsilon v_1) \cdots \text{cum}(N(t_1), i\varepsilon v_p)_{t_1 = t_1} \end{aligned} \quad (18)$$

where the summation \sum extends over all partitions v_1, \dots, v_p of $\{1, \dots, k\}$ for $p = 2, \dots, k$. Now by the induction assumption and (12), (17)

$$\begin{aligned} & \text{cum}(N(t_1), \dots, N(t_1)) \\ &= E\{N(t_1)^k\} - \sum_{i=2}^k C_{i,k} \mu^i \\ &= \sum_{i=1}^k C_{i,k} \mu^i - \sum_{i=2}^k C_{i,k} \mu^i = C_{1,k} \mu = \mu \end{aligned} \quad (19)$$

Combine (15) and (19), (14) follows.

To prove the sufficiency we need to prove for any interval $(t_1, t_2]$, $N(t_2) - N(t_1)$ is Poisson distributed and $N(t)$ has independent increments. From (14) and $E\{N(t)^k\} < \infty$, we have

$$\text{cum}(N(t), \dots, N(t)) = EN(t) \quad (20)$$

Let $\mu = EN(t)$ and substitute (20) in (18) (replacing t_1 by t), we have

$$E\{N(t)^k\} = \mu + \sum_{i=2}^k C_{i,k} \mu^i = \sum_{i=1}^k C_{i,k} \mu^i$$

which coincides with the moments of Poisson rv's given in (12).

It is well known (see Y. S. Chow 1978, p. 280) that if the characteristic function of a distribution F is analytic, the F is uniquely determined by its' moment sequence. Since the characteristic function of Poisson is analytic in the whole plane, we have that $N(t)$ is Poisson with mean $E(N(t))$.

For increments $N(t_2) - N(t_1)$ on $(t_1, t_2]$ we need only check

$$\text{cum}(N(t_2) - N(t_1), \dots, N(t_2) - N(t_1)) = E(N(t_2) - N(t_1)) \quad (21)$$

since the above argument gives that $N(t_2) - N(t_1)$ is Poisson. By (14) it is easy to see that

$$\begin{aligned}
 & \text{cum}(N(t_1) - N(t_0), \dots, N(t_i) - N(t_0)) \\
 &= \text{cum}(N(t_1), N(t_1) - N(t_0), \dots, N(t_i), N(t_i) - N(t_0)) \\
 &= \text{cum}(N(t_1), N(t_1) - N(t_0), \dots, N(t_i)) \\
 &= \text{cum}(N(t_1), N(t_1) - N(t_0), \dots, N(t_i) - N(t_0)) \\
 &= \text{cum}(N(t_1), \dots, N(t_i), N(t_i) - N(t_0)) \\
 &= EN(t_i) - EN(t_0) = E(N(t_i) - N(t_0))
 \end{aligned}$$

Here we use the multilinear property of joint cumulant.

It follows from (21) that $N(t_2) - N(t_1)$ is Poisson distributed with mean $E\{N(t_2) - N(t_1)\}$. Consequently, the characteristic function of $N(t_i) - N(t_{i-1})$ is analytic. We also know that, for rv's X_1, \dots, X_r if for each $i, i = 1, \dots, m$

$$\sum_k \frac{\theta^k}{k!} EX_i^k \quad (22)$$

has a positive radius of convergence as a power series in θ , then the joint distribution of (X_1, \dots, X_m) is determined by its cross moment $EX_1^{r_1}, \dots, X_m^{r_m}$ where r_i are nonnegative integers (see Billingsley, 1979, p. 351) for Poisson marginal (22) hold. To prove independent increments we only need to check

$$EX_1^{r_1}, \dots, X_m^{r_m} = EX_1^{r_1}, \dots, EX_m^{r_m} \quad (23)$$

for all $r_i, i=1, \dots, m$ where $X_i = N(t_i) - N(t_{i-1})$ and where $(t_{i-1}, t_i], i = 1, \dots, m$ are disjoint intervals on R^+ .

We check (23) by using Lemma 3 and induction. For $m=1$, it's trivial. For $m=2, r_1 = r_2 = 1$,

$$\begin{aligned}
 \text{Cov}(X_1, X_2) &= \text{Cov}(N(t_1) - N(t_0), N(t_2) - N(t_1)) \\
 &= \text{Cov}(N(t_1), N(t_2)) - \text{Cov}(N(t_0), N(t_2)) \\
 &\quad - \text{Cov}(N(t_1), N(t_2)) + \text{Cov}(N(t_0), N(t_1)) \\
 &= EN(t_1) - EN(t_0) - EN(t_1) + EN(t_0) = 0
 \end{aligned}$$

So $EX_1 X_2 = EX_1 EX_2$. Similarly, we have

$$\text{Cov}(X_i, X_j) = 0 \quad i \neq j$$

which means increments on disjoint intervals are uncorrelated.

By (14), we also have

$$\text{cum } (N(t_1) - N(t_0), \dots, N(t_m) - N(t_{m-1})) = 0 \quad (25)$$

which means increments on disjoint intervals have zero joint cumulant. More generally,

$$\begin{aligned} & \text{cum } (\overbrace{X_1, \dots, X_1}^{r_1}, \dots, \overbrace{X_m, \dots, X_m}^{r_m}) \\ &= 2^{r_m-1} \{EN(t_1) - EN(t_1)\} \\ &+ \left[\sum_{j=1}^m \sum_{r_j-1} - 2^{r_m-1} \right] \{EN(t_0) - EN(t_0)\} = 0 \end{aligned} \quad (26)$$

We do a simple case as example. Assume $t_0 < t_1 < t_2$,

$$\text{cum } (X_1, X_1, X_2)$$

$$\begin{aligned} &= \text{cum } (N(t_1) - N(t_0), N(t_1) - N(t_0), (N(t_2) - N(t_1))) \\ &= \text{cum } (N(t_1), N(t_1), N(t_2)) - \text{cum } (N(t_1), N(t_1), N(t_1)) \\ &- \text{cum } (N(t_1), (N(t_0), N(t_2))) + \text{cum } (N(t_1), N(t_0), N(t_1)) \\ &- \text{cum } (N(t_0), N(t_1), N(t_2)) + \text{cum } (N(t_0), N(t_1), N(t_1)) \\ &+ \text{cum } (N(t_0), N(t_0), N(t_2)) - \text{cum } (N(t_0), N(t_0), (N(t_1))) \\ &= \{EN(t_1) - EN(t_1)\} + 3 \{EN(t_0) - EN(t_0)\} = 0 \end{aligned}$$

From (26), Lemma 3 and induction (23) follows for all r_i , which imply independent increments for $N(t)$. This completes the proof.

As a by-product we have the following result.

Theorem 5. If the rv X satisfies $E|X|^n < \infty$ for all n , the X is Poisson distributed if and only if $\text{cum}(\underbrace{X, \dots, X}_m) = EX$ for all m .

The characterization here provides another example of independence characterized via uncorrelation. Test methods for a counting process being Poisson can be constructed by using this result.

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